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Effective macroscopic description for heat conduction in heterogeneous materials

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Abstract—The macroscopic thermal behaviour of heterogeneous materials is studied using the ensemble averaging technique. The non-local constitutive relations for heat conduction are derived. They relate the ensemble averaged heat flux and energy density to the ensemble averaged temperature of the medium. All the effective properties appearing in the relations are defined with help of the newly introduced, so-called microstructure functions. The asymptotic behaviour of the heterogeneous media for slowly varying average temperature fields was investigated. Possible applications of the theory are presented in two examples when analytical results could be easily obtained.

INTRODUCTION

HETEROGENEOUS materials like porous, granular media, solid suspensions, two-phase media or composites are widely used in engineering practice. The behaviour of the materials under different thermal loads is seriously influenced by their inner structure. The microstructure is usually so complicated that it forces a macroscopic (effective) approach for description of heat transfer processes in the media to be commonly adopted. In this macroscopic approach the non-homogeneous medium with step-like properties is replaced, in some averaged sense, by a continuum with constant (or continuously changing) effective properties.

Two forms of continuum models are usually utilized. In the first case, called pseudohomogeneous, the heterogeneous medium is modelled as a single phase with certain effective properties and its behaviour is described by one averaged temperature [1–3]. In the second one, called heterogeneous (or mixture), components are invisioned as forming continua coexisting at every point of the medium [4, 5]. Different effective properties and average temperatures are assigned to each of the continua which in turn mutually exchange energy.

The following mutually combined problems are often met when an effective approach is applied for continuum modelling of heat transfer processes in heterogeneous media.

(i) Formulation of energy balance equations for averaged quantities.
(ii) Closure problem for the equations understood as a need for stating relations between averaged quantities appearing in them (e.g. averaged heat flux) and averaged temperature (or averaged temperatures in the mixture models).

If the above problems are solved more or less standard analytical or numerical methods may be used to obtain a distribution of the averaged temperature in the medium and to calculate quantities derived from it. Solution of the problems is, however, very difficult (and as yet not totally solved) due to the presence of different modes of heat transport in heterogeneous media (conduction, convection, thermal radiation) as well as the complicated microstructure of the media which may even change during the process.

In the paper it was assumed that the conduction mode of heat transfer in the medium is the only one present and that the medium does not change its microgeometry with time. Thermal properties of constituents are treated as temperature independent as is usual.

Most of the recent approaches to continuum modelling of heat conduction in heterogeneous materials are limited to periodic structures [1–3, 6], unbounded domains [7, 8] or processes or slowly varying with time [9, 10]. At the same time boundary effects, manifesting themselves as drops of temperature in the vicinity of walls bounding the heterogeneous medium were experimentally observed [11, 12]. They may seriously influence the heat transfer rate, for, e.g. in chemical reactor technology especially for low ratios of particle diameter to stagnant bed dimensions. When short-duration temperature pulses are initiated in an heterogeneous medium they may lead to quite different thermal behaviour from that theoretically predicted [13, 14]. For instance, thermal diffusivity measurements of fiber-reinforced or laminated composites (e.g. by the flash method) happened to decrease with time [15]. It seems then justified to try to derive a more general theory that could take these observations potentially in account and could refer to both periodic and random media.

NOMENCLATURE

A	inclusion surface	Greek symbols	
c	volumetric specific heat	δ	Dirac's function
e	internal energy density	λ	thermal conductivity
G	Green's function	ξ	(local) coordinate vector
G_∞	Green's function for infinite medium	θ	component characteristic function
$\mathbf{k}$	wave vector, $\|\mathbf{k}\| = k = 2\pi/\Lambda$	ϕ	microstructure function of the first kind
l	characteristic microdimension	ψ	microstructure function of the second kind
L	characteristic macrodimension	Ω	volume.
$\mathbf{n}$	outwardly directed unit vector		
$\mathbf{q}$	heat flux	Other symbols	
t	time	$\langle\cdot\rangle_i$	volume average for the test inclusion
T	temperature	$\{\cdot\}$	ensemble average
v	volume fraction	$\{\cdot\}^*$	conditional ensemble average
$\mathbf{x}$	(global) coordinate vector.	∂_t	time derivative.

Note: upper indices closed in brackets indicate tensor order of a quantity.

ENSEMBLE AVERAGING AND APPROXIMATE DESCRIPTION OF MICROSTRUCTURE OF A HETEROGENEOUS MEDIUM

Methods of averaging

Two different averaging methods are broadly used in formulation of macroscopic heat conduction equation in heterogeneous materials. The most popular is volume averaging [6, 16, 17]. When this averaging procedure is applied to an arbitrary field quantity the integration is carried over the so-called representative volume Ω_e surrounding the point in question $\mathbf{x}$

$$\langle\cdot(t,\mathbf{x})\rangle = \Omega_e^{-1}\int_{\Omega_e}\cdot(t,\mathbf{x}+\mathbf{r})\,\mathrm{d}\mathbf{r}.$$

In periodic structures the point $\mathbf{x}$ is equated to a grid point $\mathbf{x}_n$ of the net while Ω_e is understood as the cell volume. Continuum values for the quantities $\langle\cdot(t,\mathbf{x})\rangle$ are derived from the discrete ones after application of the Fourier transform and truncation the obtained series by the so-called first Brillouin zone [1]. For media with no periodic structure all that is known of the averaging volume Ω_e is that its dimensions should be much greater than the microdimension related to the structure and much less than the bulk dimension of the medium.

Another technique of averaging recently became more widely accepted—ensemble averaging [5, 8, 10, 18, 19]. In this technique one strictly determined way of geometric distribution of constituents is treated as a separate configuration A—an element of the sample space. The configuration may be defined both for random or periodic distribution of constituents. Any change in position of constituents with respect to boundaries of the medium is understood as a different configuration A. The ensemble average of any function $\cdot(t,\mathbf{x}|A)$ is then defined as

$$\{\cdot(t,\mathbf{x})\} = \int\cdot(t,\mathbf{x}|A)p(A)\,\mathrm{d}A \tag{1}$$

where $p(A)$ is a probabilistic density function and $p(A)\,\mathrm{d}A$ a probabilistic measure over the sample space.

The main advantage of ensemble averaging (in contrast to volume averaging) is that operations of differentiation and ensemble averaging commutate in most cases. This follows from the appropriate theorems about differentiation of integral expressions. Moreover ensemble averaging may be used for random and (with proper interpretation) for periodic media. Boundary problems, practically insoluble with the volume averaging method, can also be handled by this method of averaging. Finally it should be noted that there exist cases when comparison of the ensemble and volume averaged quantities is possible. If a random field is defined over an infinite region of space and if it is statistically homogeneous, i.e. its ensemble average is not a function of position, then it is possible to invoke the ergodic theorem. The theorem equates the ensemble average with the volume one. Due to greater versatility of the ensemble averaging technique it will be used in the subsequent derivation of the effective heat conduction equation.

Description of microstructure

The microstructure of a heterogeneous medium is the main feature that distinguishes homogeneous and heterogeneous materials. Description of the microstructure is thus an important element of studying the effective behaviour of heterogeneous materials. It is convenient to describe any configuration A of constituents distribution in the medium with help of the so-called characteristic function θ_j associated with the jth kind of constituent and defined as

$$\theta_j(\mathbf{x}|A) = \begin{cases} 1 \text{ for } \mathbf{x} \in j\text{th kind of constituent} \\ 0 \text{ for } \mathbf{x} \notin j\text{th kind of constituent} \end{cases} \quad (2)$$

where

$$\sum_{j=1}^{N} \theta_j(\mathbf{x}|A) = 1$$

and N is the number of constituents.

Except for periodic materials, complete knowledge of the characteristic function is not practically possible. Two different ways of description of the microstructure are widespread.

In the first case the n-point correlation functions are used. They are the statistical moments of the various order of products of the characteristic function belonging to the same or different constituents of the medium [20, 21]. The correlation functions allow statistical classification of the various types of heterogeneous materials to be introduced [21]. Nevertheless they are very difficult to determine experimentally so some model correlation functions have been proposed. It is, however, difficult to state relations between these model functions and characteristic features of the microstructure of the real media.

In the second case, an approximate description of a heterogeneous medium geometry is done by stating the probability (or probabilistic density) associated with the position of different phase grains in the medium, their shape, orientation and dimensions. The probabilistic density function $p(A)$ in (1) may then be expressed as a product

$$p(A) = p_1(\mathbf{x}) \cdot p_{2/1}(A|\mathbf{x}) \cdot \ldots$$

where p_1 is the so-called 'one-particle distribution' function, $p_{2/1}$—'pair distribution' function. This way of describing the microgeometry of a heterogeneous medium, although less general than the former one, allows the influence of the characteristic features of the medium structure on the heat conduction process to be studied much more simply.

FORMULATION OF THE EFFECTIVE HEAT CONDUCTION EQUATION FOR A HETEROGENEOUS MEDIUM

The heat conduction equation for a definite distribution of constituents in volume Ω occupied by a heterogeneous medium (i.e. configuration A) may be obtained from the energy balance equation

$$-\nabla \cdot \mathbf{q}(t, \mathbf{x}|A) + q_v(t, \mathbf{x}) = \partial_t e(t, \mathbf{x}|A) \quad (3)$$

when the following constitutive relations between heat flux $\mathbf{q}$, inner energy density e and temperature T are utilized

$$-\mathbf{q}(t, \mathbf{x}|A) = \lambda(\mathbf{x}|A)\nabla T(t, \mathbf{x}|A) \quad (4)$$

$$e(t, \mathbf{x}|A) = e_0 + c(\mathbf{x}|A)T(t, \mathbf{x}|A) \quad (5)$$

where

$$\lambda(\mathbf{x}|A) = \sum_{j=1}^{N} \lambda_j \theta_j(\mathbf{x}|A), \quad c(\mathbf{x}|A) = \sum_{j=1}^{N} c_j \theta_j(\mathbf{x}|A).$$

The heat conduction equation should be completed with initial and boundary conditions. The subsequent form of the conditions was assumed

$$T(0, \mathbf{x}|A) = T_0(\mathbf{x}|A) \quad \text{in } \Omega \quad (6)$$

$$-\mathbf{q}(t, \mathbf{x}|A) \cdot \mathbf{n} + \alpha(\mathbf{x})T(t, \mathbf{x}|A) = f_A(t, \mathbf{x}) \quad \text{on } \partial\Omega. \quad (7)$$

Dirichlet, Neuman or Cauchy type boundary conditions may be obtained from (7) by proper choice of functions $\alpha(\mathbf{x})$ and $f_A(t, \mathbf{x})$. It should be stressed that these functions as well as a function describing density of heat sources in the medium in equation (3) are assumed to be independent of the medium microstructure.

By taking the ensemble average of equations (3)–(5) the following expressions were obtained

$$-\nabla \cdot \{\mathbf{q}(t, \mathbf{x})\} + q_v(t, \mathbf{x}) = \partial_t \{e(t, \mathbf{x})\} \quad (8)$$

$$-\{\mathbf{q}(t, \mathbf{x})\} = \{\lambda(\mathbf{x})\nabla T(t, \mathbf{x})\} \quad (9)$$

$$\{e(t, \mathbf{x})\} = e_0 + \{c(\mathbf{x})T(t, \mathbf{x})\} \quad (10)$$

together with the initial and boundary conditions:

$$\{T(0, \mathbf{x})\} = \{T_0(\mathbf{x})\} \quad \text{in } \Omega \quad (11)$$

$$-\{\mathbf{q}(t, \mathbf{x})\} \cdot \mathbf{n} + \alpha(\mathbf{x})\{T(t, \mathbf{x})\} = f_A(\mathbf{x}) \quad \text{on } \partial\Omega. \quad (12)$$

The set of the equations (8)–(12) may only be solved when relations between average heat flux $\{\mathbf{q}\}$, inner energy $\{e\}$ and the average temperature $\{T\}$ are given, that is, the previously mentioned 'closure problem' for energy balance equation is solved. These relations could, however, be derived from (9) and (10) when dependence of temperature $T(t, \mathbf{x}|A)$ on the ensemble averaged temperature $\{T(t, \mathbf{x})\}$ is known. The latter relation may be obtained according to the following reasoning.

The energy balance equation (3) can be rearranged using equations (4) and (5) to

$$\nabla \cdot \lambda_0 \nabla T(t, \mathbf{x}|A) + [\nabla \cdot \lambda'(\mathbf{x}|A)\nabla T(t, \mathbf{x}|A) + q_v(t, \mathbf{x}) - c(\mathbf{x}|A)\partial_t T(t, \mathbf{x}|A)] = 0 \quad (13)$$

where λ_0 is a constant reference thermal conductivity and

$$\lambda'(\mathbf{x}|A) = \lambda(\mathbf{x}|A) - \lambda_0. \quad (14)$$

Equation (13) can be formally solved by regarding the second term on the left-hand side as an imaginary heat source

$$T(t,\mathbf{x}|A) = -\int_\Omega \nabla G(\mathbf{x},\mathbf{x}')\cdot\lambda'(\mathbf{x}|A)\nabla T(t,\mathbf{x}'|A)\,\mathrm{d}\mathbf{x}' - \int_\Omega G(\mathbf{x},\mathbf{x}')[q_v(t,\mathbf{x}') - c(\mathbf{x}'|A)\partial_t T(t,\mathbf{x}'|A)]\,\mathrm{d}\mathbf{x}' - \int_{\partial\Omega} G(\mathbf{x},\mathbf{x}')\mathbf{q}(t,\mathbf{x}'|A)\cdot\mathbf{n}\,\mathrm{d}\mathbf{x}' - \int_{\partial\Omega}\lambda_0\nabla G(\mathbf{x},\mathbf{x}')\cdot\mathbf{n}T(t,\mathbf{x}'|A)\,\mathrm{d}\mathbf{x}' \quad (15)$$

where $\mathbf{n}$ is the external unit vector to the surface $\partial\Omega$.

The Green's function $G(\mathbf{x},\mathbf{x}')$ appearing in equation (15) is a steady-state one defined by

$$\nabla\cdot\lambda_0\nabla G(\mathbf{x},\mathbf{x}')+\delta(\mathbf{x},\mathbf{x}') = 0 \quad \text{in}\ \Omega$$
$$\lambda_0\nabla G(\mathbf{x},\mathbf{x}')\cdot\mathbf{n}+\alpha(\mathbf{x})G(\mathbf{x},\mathbf{x}') = 0 \quad \text{on surface}\ \partial\Omega \quad (16)$$

where $\delta(\mathbf{x},\mathbf{x}')$ is the Dirac delta function.

By taking the ensemble average of equation (15), subtracting the obtained expression from the former one and taking into account equation (16) the following equation may be written

$$T(t,\mathbf{x}|A) = \{T(t,\mathbf{x})\} - \int_\Omega \nabla G(\mathbf{x},\mathbf{x}')\cdot[\lambda'(\mathbf{x}'|A)\nabla T(t,\mathbf{x}'|A) - \{\lambda'(\mathbf{x}')\nabla T(t,\mathbf{x}')\}]\,\mathrm{d}\mathbf{x}' - \int_\Omega G(\mathbf{x},\mathbf{x}')[c(\mathbf{x}'|A)\partial_t T(t,\mathbf{x}'|A) - \{c(\mathbf{x}')\partial_t T(t,\mathbf{x}')\}]\,\mathrm{d}\mathbf{x}'. \quad (17)$$

We now look for such a solution of the above equation that would enable influence of the ensemble averaged temperature $\{T(t,\mathbf{x})\}$ on temperature $T(t,\mathbf{x}|A)$ to be effectively separated from fluctuations of temperature related to the local change of thermal properties of the medium. The solution may be formally cast in the form

$$T(t,\mathbf{x}|A) = \{T(t,\mathbf{x})\} + \int_\Omega \boldsymbol{\phi}(\mathbf{x},\mathbf{x}'|A)\cdot\nabla\{T(t,\mathbf{x}')\}\,\mathrm{d}\mathbf{x}' + \int_\Omega\int_0^t \psi(t,t',\mathbf{x},\mathbf{x}'|A)\,\partial_{t'}\{T(t',\mathbf{x}')\}\,\mathrm{d}t'\,\mathrm{d}\mathbf{x}'. \quad (18)$$

Two unknown functions: steady one $\boldsymbol{\phi}(\mathbf{x},\mathbf{x}'|A)$ and transient $\psi(t,t',\mathbf{x},\mathbf{x}'|A)$ which appear in equation (18) were, in the subsequent reasoning, referred as the microstructure functions of the first and second kind, respectively. By introducing equation (18) into equation (17) the following integro-differential equations for the microstructure functions were obtained

$$\boldsymbol{\phi}(\mathbf{x},\mathbf{x}'|A) = -\int_\Omega \nabla G(\mathbf{x},\mathbf{y})\cdot[\lambda'(\mathbf{y}|A)(\delta(\mathbf{y},\mathbf{x}')\mathbb{1} + \nabla\boldsymbol{\phi}(\mathbf{y},\mathbf{x}'|A)) - \{\lambda'(\mathbf{y})(\delta(\mathbf{y},\mathbf{x}')\mathbb{1} + \nabla\boldsymbol{\phi}(\mathbf{y},\mathbf{x}'))\}]\,\mathrm{d}\mathbf{y}. \quad (19)$$

$$\psi(t,t',\mathbf{x},\mathbf{x}'|A) = -\int_\Omega \nabla G(\mathbf{x},\mathbf{y})\cdot[\lambda'(\mathbf{y}|A)\nabla\psi(t,t',\mathbf{y},\mathbf{x}'|A) - \{\lambda'(\mathbf{y})\nabla\psi(t,t',\mathbf{y},\mathbf{x}')\}]\,\mathrm{d}\mathbf{y}' - \int_\Omega G(\mathbf{x},\mathbf{y})[c(\mathbf{y}|A)(\delta(\mathbf{y},\mathbf{x}') - \nabla\cdot\boldsymbol{\phi}(\mathbf{y},\mathbf{x}'|A)) - \{c(\mathbf{y})(\delta(\mathbf{y},\mathbf{x}') - \nabla\cdot\boldsymbol{\phi}(\mathbf{y},\mathbf{x}'))\}]\,\mathrm{d}\mathbf{y}\,\delta(t,t') - \int_\Omega G(\mathbf{x},\mathbf{y})[c(\mathbf{y}|A)\,\partial_t\psi(t,t',\mathbf{y},\mathbf{x}'|A) - \{c(\mathbf{y})\,\partial_t\psi(t,t',\mathbf{y},\mathbf{x}')\}]\,\mathrm{d}\mathbf{y} \quad (20)$$

where

$$\psi(t,t'=t,\mathbf{x},\mathbf{x}'|A) = 0. \quad (21)$$

It should be noted that the microstructure functions $\boldsymbol{\phi}$, ψ are not dependent on the average temperature of the medium.

The expression (18) is then substituted into equations (9) and (10). This leads to the following relations between the ensemble averaged heat flux or internal energy density and temperature

$$-\{\mathbf{q}(t,\mathbf{x})\} = \int_\Omega \boldsymbol{\lambda}_{\mathrm{ef}}(\mathbf{x},\mathbf{x}')\cdot\nabla\{T(t,\mathbf{x}')\}\,\mathrm{d}\mathbf{x}' + \int_\Omega\int_0^t \mathbf{v}_{\mathrm{ef}}(t,t',\mathbf{x},\mathbf{x}')\,\partial_{t'}\{T(t',\mathbf{x}')\}\,\mathrm{d}t'\,\mathrm{d}\mathbf{x}' \quad (22)$$

$$\{e(t,\mathbf{x})\} = e_0 + c_{\mathrm{ef}}\{T\} + \int_\Omega \boldsymbol{\chi}_{\mathrm{ef}}(\mathbf{x},\mathbf{x}')\cdot\nabla\{T(t,\mathbf{x}')\}\,\mathrm{d}\mathbf{x}' + \int_\Omega\int_0^t \mu_{\mathrm{ef}}(t,t',\mathbf{x},\mathbf{x}')\,\partial_{t'}\{T(t',\mathbf{x}')\}\,\mathrm{d}t'\,\mathrm{d}\mathbf{x}'. \quad (23)$$

The quantities $\boldsymbol{\lambda}_{\mathrm{ef}}$, $\mathbf{v}_{\mathrm{ef}}$, c_{ef}, $\boldsymbol{\chi}_{\mathrm{ef}}$ and μ_{ef} are defined as

$$\boldsymbol{\lambda}_{\mathrm{ef}}(\mathbf{x},\mathbf{x}') = \{\lambda(\mathbf{x})[\mathbb{1}\delta(\mathbf{x},\mathbf{x}')+\nabla\boldsymbol{\phi}(\mathbf{x},\mathbf{x}')]\},$$
$$\mathbf{v}_{\mathrm{ef}}(t,t',\mathbf{x},\mathbf{x}') = [\lambda(\mathbf{x})\nabla\psi(t,t',\mathbf{x},\mathbf{x}')\},$$
$$c_{\mathrm{ef}}(\mathbf{x}) = \{c(\mathbf{x})\},$$
$$\boldsymbol{\chi}_{\mathrm{ef}}(\mathbf{x},\mathbf{x}') = \{c(\mathbf{x})\boldsymbol{\phi}(\mathbf{x},\mathbf{x}')\},$$
$$\mu_{\mathrm{ef}}(t,t',\mathbf{x},\mathbf{x}') = \{c(\mathbf{x})\psi(t,t',\mathbf{x},\mathbf{x}')\}. \quad (24)$$

The relations (22) and (23) are non-local ones. This means that, in contrast to local constitutive relations (e.g. of Fourier law type), the ensemble averaged heat flux $\{\mathbf{q}\}$ and energy density $\{e\}$ depend not only on the ensemble averaged temperature $\{T\}$ (or its derivatives) in the considered point $\mathbf{x}$ and moment of time t but on a distribution of the average temperature in the whole volume embraced by the heterogeneous material and changes of the distribution from the starting point of a thermal process up to the considered time t, i.e. are history dependent.

Non-local relations between physical quantities were known in the past. They were, for instance, proposed in the elastostatics, viscoelasticity or electrodynamics of dispersive media. In contrast to them, however, in the present case the exact expressions for estimation of material functions appearing in equations (22)–(24) were given. The additional virtue of the presented theory is also a possibility of calculation of local values of temperature $T(t, \mathbf{x}|A)$ for a definite distribution of constituents A from the known values of the ensemble averaged temperature $\{T(t, \mathbf{x})\}$ and the proper microstructure functions.

When the conditions stated by Eringen in ref. [24] are fulfilled the relations (22) and (23) may become the constitutive equations for the effective pseudo-homogeneous medium. Then the material functions λ_{ef}, ν_{ef}, c_{ef}, χ_{ef} and μ_{ef} could be treated as the effective thermal properties of the heterogeneous medium.

If the discussed relations are introduced into the ensemble averaged energy balance equation (8) the effective heat conduction equation will be obtained.

FORMULATION OF THE EFFECTIVE HEAT CONDUCTION EQUATION FOR SLOWLY VARYING AVERAGE TEMPERATURE FIELDS

In the most types of heterogeneous media it is possible to find out a certain characteristic length l related to variation of such local properties of the medium as its thermal conductivity or specific heat. This characteristic length may be, for instance, a distance between inclusions in periodic materials, a dimension of a grain or cell in granular or polycrystalline materials, characteristic radius of correlation in the statistical description of the medium microgeometry. The length l will be subsequently called the microdimension of a heterogeneous medium.

On the basis of the microdimension a local coordinate system can be introduced. In order to do this a point $\mathbf{x}_l^*$ was distinguished from the whole set of characteristic points $\mathbf{x}_i^*(A)$ used for geometrical description of the microstructure. The point $\mathbf{x}_l^*$ has a unique feature of being the nearest to the considered point $\mathbf{x}$ of the medium (Fig. 1). The local coordinate system combined with this point is then defined as

$$\xi(A) = [\mathbf{x}-\mathbf{x}_l^*(A)]/l. \qquad (25)$$

The quantities related to heat conduction in a heterogeneous material may be divided into three groups.

(1) Dependent only on configuration A, e.g. $\lambda(\mathbf{x}|A)$, $c(\mathbf{x}|A)$.

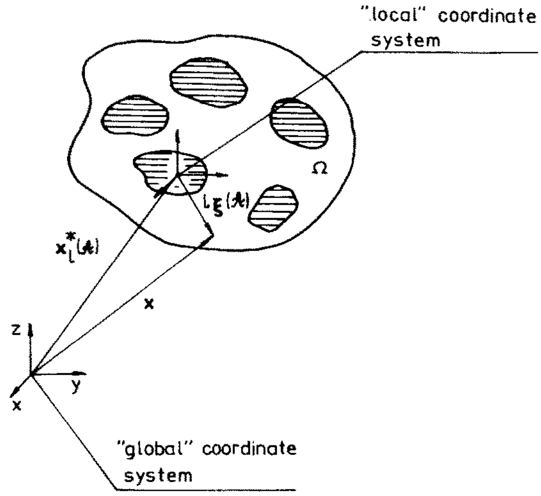

FIG. 1. Relation between 'local' ξ, and 'global' $\mathbf{x}$, system of coordinates.

(2) Dependent both on configuration A and on a position with respect to boundaries of the medium, e.g. $T(t, \mathbf{x}|A)$, $\mathbf{q}(t, \mathbf{x}|A)$, $e(t, \mathbf{x}|A)$.
(3) Independent of configuration, e.g. e_0, λ_0, $q_v(\mathbf{x})$.

Every field quantity dependent solely on microstructure (i.e. on configuration A) may be rewritten in the form

$$\cdot(\mathbf{x}|A) = \cdot(\xi|A)$$

while this being a function not only of a configuration A but also of position with respect to boundaries may be presented as

$$\cdot(t, \mathbf{x}|A) = \cdot(t, \xi, \mathbf{x}|A).$$

The gradient of a quantity belonging to the first group may be expressed as

$$\nabla\cdot(\mathbf{x}|A) = l^{-1}\nabla_l\cdot(\xi|A).$$

while that belonging to the second group can be cast in the following form

$$\nabla\cdot(t, \mathbf{x}|A) = \nabla\cdot(t, \xi, \mathbf{x}|A) = \nabla_g\cdot(t, \xi, \mathbf{x}|A) + l^{-1}\nabla_l\cdot(t, \xi, \mathbf{x}|A)$$

where ∇_l denotes operation of differentiation with respect to the local system of coordinates ξ, ∇_g with respect to global system of coordinates $\mathbf{x}$.

If changes in temperature are relatively slow, both in space and time, a difference between a local value of temperature $T(t, \mathbf{x}|A)$ and the ensemble averaged temperature $\{T(t, \mathbf{x})\}$ is small. The difference is given by the second and third term on the right-hand side of equation (18). The equation may be then presented in a slightly transformed form

$$T(t,\mathbf{x}|A) = \{T(t,\mathbf{x})\} + l\int_\Omega \boldsymbol{\phi}(\boldsymbol{\xi},\mathbf{x},\mathbf{x}'|A)/l \cdot \nabla\{T(t,\mathbf{x}')\}\,\mathrm{d}\mathbf{x}')\}\,\mathrm{d}\mathbf{x}' + l^2\int_\Omega\int_0^t \psi(t,t',\boldsymbol{\xi},\mathbf{x},\mathbf{x}'|A)/l^2 \times \partial_{t'}\{T(t',\mathbf{x}')\}\,\mathrm{d}t'\,\mathrm{d}\mathbf{x}'. \quad (26)$$

This form of the equation (26) suggests application of a multipole expansion technique to be used in order to obtain a simplified version of the relations (22) and (23). The following expansions for the microstructure functions were assumed

$$\boldsymbol{\phi}(\boldsymbol{\xi},\mathbf{x},\mathbf{x}'|A)/l = \boldsymbol{\phi}_1(\boldsymbol{\xi},\mathbf{x}|A)\cdot\delta(\mathbf{x},\mathbf{x}') - l\boldsymbol{\phi}_2^{(2)}(\boldsymbol{\xi},\mathbf{x}|A)\cdot\nabla_g\delta(\mathbf{x},\mathbf{x}') + \cdots \quad (27)$$

$$\psi(t,t',\boldsymbol{\xi},\mathbf{x},\mathbf{x}'|A)/l^2 = \psi_{20}(\boldsymbol{\xi},\mathbf{x}|A)\delta(\mathbf{x},\mathbf{x}')\delta(t,t') - l\boldsymbol{\psi}_{30}(\boldsymbol{\xi},\mathbf{x}|A)\cdot\nabla_g\delta(\mathbf{x},\mathbf{x}')\delta(t,t') + \cdots \quad (28)$$

The equations (27) and (28) may be interpreted as expansions in a series of a small parameter l/L where L is a characteristic macrodimension connected with changes of temperature for a reference, homogeneous material of shape, dimensions and thermal boundary conditions identical to these of the heterogeneous one.

When the expansions (27), (28) had been introduced into equation (26) the following expression for the relation between local and the ensemble averaged temperature was obtained

$$T(t,\mathbf{x}|A) = \{T(t,\mathbf{x})\} + l\boldsymbol{\phi}_1(\boldsymbol{\xi},\mathbf{x}|A)\cdot\nabla_g\{T(t,\mathbf{x})\} + l^2[\boldsymbol{\phi}_2^{(2)}(\boldsymbol{\xi},\mathbf{x}|A):\nabla_g^{(2)}\{T(t,\mathbf{x})\} + \psi_{20}(\boldsymbol{\xi},\mathbf{x}|A)\partial_t\{T(t,\mathbf{x})\}] + O(l^3). \quad (29)$$

Similarly from (22) and (23) the following relations binding the ensemble averaged heat flux, energy density and temperature were derived

$$-\{\mathbf{q}(t,\mathbf{x})\} = \boldsymbol{\lambda}_{2\mathrm{ef}}^{(2)}(\mathbf{x})\cdot\nabla_g\{T(t,\mathbf{x})\} + l\boldsymbol{\lambda}_{3\mathrm{ef}}^{(3)}(\mathbf{x})\cdot\nabla_g^{(2)}\{T(t,\mathbf{x})\} + l^2[\boldsymbol{\lambda}_{4\mathrm{ef}}^{(4)}(\mathbf{x}):\nabla_g^{(3)}\{T(t,\mathbf{x})\} + \boldsymbol{v}_{2\mathrm{ef}}^{(3)}(\mathbf{x}):\nabla_g^{(2)}\partial_t\{T(t,\mathbf{x})\}] + \cdots \quad (30)$$

$$\{e(t,\mathbf{x})\} = e_0 + c_{\mathrm{ef}}(\mathbf{x})\{T(t,\mathbf{x})\} + l\boldsymbol{\chi}_{1\mathrm{ef}}(\mathbf{x})\cdot\nabla_g\{T(t,\mathbf{x})\} + l^2[\boldsymbol{\chi}_{2\mathrm{ef}}^{(2)}(\mathbf{x}):\nabla_g^{(2)}\{T(t,\mathbf{x})\} + \mu_{2\mathrm{ef}}(\mathbf{x})\partial_t\{T(t,\mathbf{x})\}] + \cdots \quad (31)$$

The material functions appearing in the above expressions are defined as

$$\boldsymbol{\lambda}_{2\mathrm{ef}}^{(2)}(\mathbf{x}) = \{\lambda(\mathbf{x})(\mathbb{1} + \nabla_1\boldsymbol{\phi}_1(\mathbf{x}))\}$$

$$\boldsymbol{\lambda}_{3\mathrm{ef}}^{(3)}(\mathbf{x}) = \{\lambda(\mathbf{x})(\mathbb{1}\boldsymbol{\phi}_1(\mathbf{x}) + \nabla_1\boldsymbol{\phi}_2^{(2)}(\mathbf{x}))\}$$

$$\boldsymbol{\lambda}_{4\mathrm{ef}}^{(4)}(\mathbf{x}) = \{\lambda(\mathbf{x})(\mathbb{1}\boldsymbol{\phi}_2^{(2)}(\mathbf{x}) + \nabla_1\boldsymbol{\phi}_3^{(3)}(\mathbf{x}))\} \quad (32)$$

$$\boldsymbol{v}_{2\mathrm{ef}}^{(2)}(\mathbf{x}) = \{\lambda(\mathbf{x})(\mathbb{1}\psi_{20}(\mathbf{x}) + \nabla_1\boldsymbol{\psi}_{30}(\mathbf{x}))\}$$

$$\boldsymbol{\chi}_{1\mathrm{ef}}(\mathbf{x}) = \{c(\mathbf{x})\boldsymbol{\phi}_1(\mathbf{x})\},$$

$$\boldsymbol{\chi}_{2\mathrm{ef}}^{(2)}(\mathbf{x}) = \{c(\mathbf{x})\boldsymbol{\phi}_2^{(2)}(\mathbf{x})\}$$

$$\mu_{2\mathrm{ef}}(\mathbf{x}) = \{c(\mathbf{x})\psi_{20}(\mathbf{x})\}.$$

The non-local relations, given by equations (22) and (23), are thus reduced to the local form, given by equations (30) and (31), for the case of temperature slowly varying in space and time. They are linear but in contrast to simple expressions valid for homogeneous media they have much more complicated form. Temperature gradients of order higher than one and time derivatives of temperature are included in them. The remanent reminding of their non-local origin is a characteristic microscale parameter l.

If a ratio of the parameter l to the characteristic length associated with changes of temperature is very small, terms starting from second order in (30) and third order in (31) may be neglected and then the analogical relations to those for homogeneous media are obtained.

If a symmetrical (from a statistical point of view) distribution of constituents is present in a heterogeneous medium, and for location far from walls bounding the medium the functions $\boldsymbol{\lambda}_{3\mathrm{ef}}^{(3)}$ in equation (30) and $\boldsymbol{\chi}_{1\mathrm{ef}}$ in equation (31) take on zero values. The discussed functions $\boldsymbol{\lambda}_{2\mathrm{ef}}^{(2)}$, $\boldsymbol{\lambda}_{4\mathrm{ef}}^{(4)}$, $\boldsymbol{v}_{2\mathrm{ef}}^{(2)}$, c_{ef}, $\boldsymbol{\chi}_{2\mathrm{ef}}^{(2)}$ and $\mu_{2\mathrm{ef}}$ can then be treated as the effective properties of the heterogeneous medium.

In order to obtain equations for the unknown functions $\phi_n^{(n)}$, $\psi_{n0}^{(n-2)}$ which appear in definitions of the effective properties, expansions (27) and (28) were substituted into equations (19) and (20). This leads to the following equations for these functions

$$\phi_n^{(n)} = -\int_\Omega \nabla_1\mathbf{G}_1\cdot[\lambda'(\mathbb{1}\phi_{(n-1)}^{(n-1)} + \nabla_1\phi_{(n)}^{(n)}) - \{\lambda'(\mathbb{1}\phi_{(n-1)}^{(n-1)} + \nabla_1\phi_{(n)}^{(n)})\}]\,\mathrm{d}\boldsymbol{\xi}' \quad (33)$$

$$\psi_{n0}^{(n-2)} = -\int \nabla_1 G_1\cdot[\lambda'(\mathbb{1}\psi_{(n-3)}^{(n-3)} + \nabla_1\psi_{(n-2)}^{(n-2)}) - \{\lambda'(\mathbb{1}\psi_{(n-3)}^{(n-3)} + \nabla_1\psi_{(n-2)}^{(n-2)})\}]\,\mathrm{d}\boldsymbol{\xi}' - \int_\Omega G_1[c\phi_{(n-2)}^{(n-2)} - \{c\phi_{(n-2)}^{(n-2)}\}]\,\mathrm{d}\boldsymbol{\xi}'. \quad (34)$$

where $G_1(\boldsymbol{\xi},\boldsymbol{\xi}')$ is a proper Green's function expressed in the local system of coordinates.

EXAMPLES OF APPLICATION OF THE THEORY

The presented theory may be used to study different phenomena associated with heat conduction in heterogeneous media such as, e.g. thermal behaviour of the media for abrupt changes in environmental conditions and boundary effects. It can be also applied for a priori estimation of effective properties of non-homogeneous materials. Two such examples have been presented for illustration below.

The non-locality of the effective thermal conductivity

Let us consider a two-component material containing spherical inclusions of radius R and thermal conductivity λ_i distributed chaotically in a matrix of thermal conductivity λ_m. The inclusions are distributed randomly in such a way that the medium may be treated as the statistically homogeneous and isotropic in the sense proposed by Kröner [21]. The statistical homogeneity means that all positions of centres of spheres have equal probability independently of the other stochastic parameters describing the medium microstructure.

The distribution of neighbour spheres with respect to a single, selected sphere in the medium is characterized by a 'pair distribution function' $p_{2/1}$ of the form given by Hashin [22]. For this, so-called 'sphere assemblage model' the probability function $p_{2/1}$ is given by the expression

$$p_{2/1} = \begin{cases} 0 & \text{for} \quad |\mathbf{x}_2^* - \mathbf{x}_1^*| < l \\ \Omega^{-1} & \text{for} \quad |\mathbf{x}_2^* - \mathbf{x}_1^*| \geqslant l \end{cases} \tag{35}$$

where $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are vectors of positions of centre of two neighbour spheres, $l = R/v_i^{1/3}$ and v_i is volume fraction of spheres. Let us additionally assume that the heterogeneous medium can be treated as infinite not to consider complications caused by presence of boundary effects.

The infinite Fourier transform has been subsequently applied to a definition of the effective thermal conductivity λ_{ef} in equation (24) and the equation (19) for the microstructure function $\boldsymbol{\phi}(\mathbf{x}|A)$. This leads to

$$\tilde{\lambda}_{ef}(\mathbf{x}, \mathbf{k}) = \lambda_m + \lambda_i' v_i (\mathbb{1} + \theta_i \nabla \tilde{\boldsymbol{\phi}}(\mathbf{x}, \mathbf{k})/v_i), \tag{36}$$

$$\tilde{\boldsymbol{\phi}}(\mathbf{x}, \mathbf{k}|A) = -\int_{\Omega} \nabla G_\infty(\mathbf{x}, \mathbf{y}) \cdot [\lambda'(\mathbf{y}|A)(\mathbb{1} \exp(i\mathbf{k} \cdot \mathbf{y}) + \nabla \tilde{\boldsymbol{\phi}}(\mathbf{y}, \mathbf{k}|A)) - \{\lambda'(\mathbf{y})(\mathbb{1} \exp(i\mathbf{k} \cdot \mathbf{y}) + \nabla \tilde{\boldsymbol{\phi}}(\mathbf{y}, \mathbf{k}))\}] \, \mathrm{d}\mathbf{y} \tag{37}$$

where $\lambda'(\mathbf{y}) = \lambda(\mathbf{y}) - \lambda_m$, $\lambda_i' = \lambda_i - \lambda_m$ and $G_\infty(\mathbf{x}, \mathbf{y})$ is the retarded Green function tending to zero at large distances, also, for the sake of clarity, quantities in the Fourier space have been marked by a tilde.

It is advantageous to seek a solution of equation (37) for an inclusion centred at the origin of the coordinate system. So for an arbitrary position of inclusion centre, e.g. at $\mathbf{x}^*$, the micro structure function $\tilde{\boldsymbol{\phi}}$ may be written in the form

$$\tilde{\boldsymbol{\phi}}(\mathbf{x}, \mathbf{k}|A) = \tilde{\boldsymbol{\phi}}(\mathbf{r}, \mathbf{k}|A) \exp(i\mathbf{k} \cdot \mathbf{x}^*) \tag{38}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}^*$. Then noting the statistical homogeneity of the heterogeneous medium the ensemble average in definition (36) can be expressed as

$$\{\theta_i(\mathbf{x}) \nabla \tilde{\boldsymbol{\phi}}(\mathbf{x}, \mathbf{k})\} = v_i \langle \{\nabla \tilde{\boldsymbol{\phi}}(\mathbf{r}, \mathbf{k})\}^* \exp(-i\mathbf{k} \cdot \mathbf{r}) \rangle_i \tag{39}$$

where $\langle \cdot \rangle_i$ is understood as the volume averaging over the considered inclusion and the symbol $\{\cdot\}^*$ denotes the conditional average defined as

$$\{\cdot(\mathbf{x}, \mathbf{k})\}^* = \int \cdot(\mathbf{x}, \mathbf{k}|A) p(A|\mathbf{x}^*) \, \mathrm{d}(A|\mathbf{x}^*)$$

where $p(A|\mathbf{x}^*) \, \mathrm{d}(A|\mathbf{x}^*)$ is meant as the conditional probability for the case when the position $\mathbf{x}^*$ of the test inclusion is fixed.

For the considered spherical shape of inclusions randomly distributed in the matrix, the solution of equation (37) was assumed in the form

$$\tilde{\boldsymbol{\phi}}(\mathbf{r}, \mathbf{k}|A) = \tilde{\mathbf{E}}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}) + \sum_{n=0}^{\infty} \tilde{C}_n(\mathbf{k}) r^n P_n(\cos \gamma_{rk}) \tag{40}$$

where P_n are Legendre polynomials of order n and γ_{rk} is treated as an angle between the $\mathbf{r}$ and $\mathbf{k}$ vectors. The above expression, after taking into account equation (38), was introduced into equation (37) which together with the assumed form of the pair distribution function $p_{2/1}$ allowed the coefficients $\mathbf{E}(\mathbf{k})$ and $C_n(\mathbf{k})$ to be easily obtained. Then substituting these coefficients into equations (39) and (36) the following expression for the effective thermal conductivity could be written

$$\tilde{\lambda}_{ef}(k)/\lambda_m = 1 + \frac{\sigma'(v_i + F)}{1 + \sigma'(1 - v_i) + F} \tag{41}$$

where $\sigma' = (\lambda_i - \lambda_m)/\lambda_m$,

$$F = \sum_{n=0}^{\infty} \frac{3\sigma' v_i (n+1)(2n+1)}{\sigma' + (2n+1)/n} \left(\frac{j_n(klv_i^{1/3})}{klv_i^{1/3}} \right)^2$$

and j_n denotes the spherical Bessel function of the first kind and nth order.

The function λ_{ef}/λ_m, calculated from (41), was plotted in Fig. 2. The modulus of the wave vector $\mathbf{k}$ may be understood as being inversely proportional to the thermal wavelength $\Lambda - k = 2\pi/\Lambda$. The value of Λ is related to the external action of the environment on the heterogeneous medium and as such may be equalized to characteristic macrodimension L of the medium. By studying the behaviour of $\tilde{\lambda}_{ef}/\lambda_m$ with l/Λ it may be concluded that the maximal values of $\tilde{\lambda}_{ef}$ correspond to the ratio of l/Λ tending to zero. For this special case the effective thermal conductivity assumes values consistent with the well-known expression given originally by Maxwell [25]. This can be easily proved by making use of the asymptotic relation

$$\lim_{y \to 0} j_n(y)/y = 1/3\delta_{1n}.$$

For the increasing values of l/Λ the effective thermal conductivity is thus diminishing. For the case when $l/\Lambda \Rightarrow \infty$ the effective thermal conductivity attains the following limit

$$\tilde{\lambda}_{ef}/\lambda_m = [v_i/\sigma + (1 - v_i)]^{-1} \tag{42}$$

where $\sigma = \lambda_i/\lambda_m$. This may be proved when another

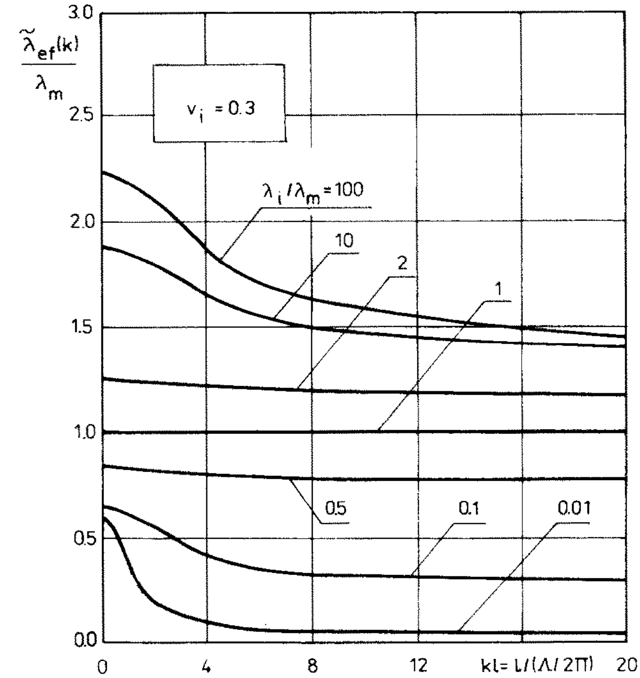

FIG. 2. The effective thermal conductivity λ_{ef} for a model of the statistically isotropic composite with spherical inclusions randomly distributed in a matrix as a function of a ratio between microdimension l and macrodimension $L = 2\pi/\Lambda$.

asymptotic relation valid for the spherical Bessel functions

$$\lim_{y\to 0} j_n(y)/y = 0$$

would be utilized.

The expression (42) is identical with a formula for λ_{ef} of a laminated composite with heat flow perpendicular to the composite layers. It should be noted that the asymptotic relation of the type given by equation (42), for the statistically isotropic heterogeneous media of an arbitrary microstructure and small differences between thermal conductivities of components, had been previously foreseen by Beran and McCoy [18] after application of the perturbation method. The similar relation for the effective thermal conductivity of heterogeneous media was also obtained by Diener and Käseberg [8] within the frame of the self-consistent approximation but using totally different method. This may serve as further evidence that the presented results are correct.

Dependence of the effective thermal conductivity on the parameter l/Λ could be interpreted in the following way. For the position in space (or moments of time) where variation in the ensemble averaged temperature $\{T\}$ are great enough that gradients of $\{T\}$ of an order higher than one could not be treated as negligible the heterogeneous material behaves as if the effective thermal conductivity was smaller than that obtained from measurements carried out in steady state. This kind of the effect had been experimentally observed by Antoniszyn [13] in measurements leading to a determination of the cooling rate for a heated calorimeter made of copper abruptly placed in a granular bed of spherical glass beads. If the bead diameter had been increased (corresponding to increase of the ratio l/Λ) it would cause a decrease in the calorimeter cooling rate for a period of approximately a few minutes in comparison to a bed made of smaller diameter beads.

The effective thermal conductivity for a two-phase composite with a small volume fraction of inclusions of arbitrary shape

Let us consider a two-component material made of a matrix of thermal conductivity λ_m and randomly distributed inclusions of an arbitrary shape and thermal conductivity λ_i. In a similar way as in the previous example the medium is to be treated as infinite so not to consider complications introduced by the presence of material boundaries. Moreover, let us assume that the ensemble averaged temperature does not substantially vary in space so that the asymptotic form of the constitutive relations $\{q\}$ and $\{T\}$, consistent with Fourier law of heat conduction in homogeneous media, is valid. This corresponds to retaining the first term in the expansion given by equation (30). According to the definition (32) the expression for the effective thermal conductivity may be then cast in the form

$$\lambda_{ef} = \{\lambda(\mathbb{1}+\nabla_1\boldsymbol{\phi}_1)\} = \lambda_m\mathbb{1} + \lambda' v_1[\mathbb{1}+\{\langle\nabla_1\boldsymbol{\phi}_1\rangle_i\}^*] \quad (43)$$

where $\lambda' = \lambda_i - \lambda_m$ and $\langle\cdot\rangle_i$ denotes the operation of averaging over volume of the selected, test inclusion for a definite shape, space orientation and location of the neighbour inclusions with respect to it. The symbol $\{\cdot\}^*$ in equation (43) is understood as the conditional statistical averaging over the whole set of possible shapes, orientations and configurations of the neighbour inclusions.

In order to calculate λ_{ef} microstructure function $\boldsymbol{\phi}_1$ is needed, which in accordance with equation (33), is a solution of the equation

$$\boldsymbol{\phi}_1(\xi, \mathbf{x}|A) = -\int_\Omega \nabla_1 G_1 \cdot [\lambda'(\mathbb{1}+\nabla_1\boldsymbol{\phi}_1) - \{\lambda'(\mathbb{1}+\nabla_1\boldsymbol{\phi}_1)\}]\,\mathrm{d}\Omega. \quad (44)$$

Both equations (43) and (44) include the expression $(\mathbb{1}+\nabla_1\boldsymbol{\phi}_1)$ so it is convenient to introduce an auxiliary function

$$\hat{\boldsymbol{\phi}}_1 = \xi + \boldsymbol{\phi}_1 \quad (45)$$

which can substantially simplify the form of these equations.

In equation (43) the value of $\nabla_1\boldsymbol{\phi}_1$ averaged over volume of the test inclusion is of interest. So with the centre of this inclusion a local coordinate system ξ had been bound. The Green's theorem was then used to change the volume to the surface integrals. This allowed equation (44) to be written in the following form

$$\hat{\boldsymbol{\phi}}_1(\xi|A) + \lambda_i' \int_A G_1(\xi,\boldsymbol{\eta})\nabla_1\hat{\boldsymbol{\phi}}_1(\boldsymbol{\eta}|A)\cdot\mathbf{n}\,\mathrm{d}\boldsymbol{\eta} + \sum_{j=2}^{\infty} \lambda_i' \int_{A_j} G_1(\xi,\boldsymbol{\eta})\nabla_1\hat{\boldsymbol{\phi}}_1(\boldsymbol{\eta}|A)\cdot\mathbf{n}\,\mathrm{d}\boldsymbol{\eta}$$
$$= \xi + \lambda_i' v_i \left\{ \int_A G_1(\xi,\boldsymbol{\eta})\langle\nabla_1\hat{\boldsymbol{\phi}}_1(\boldsymbol{\eta})\rangle_i \cdot\mathbf{n}\,\mathrm{d}\boldsymbol{\eta}\right\}^*. \quad (46)$$

The first two terms on the left-hand side of the above equation refer to the test inclusion while the third term represents the influence of the other inclusions on the test one. The second term on the right-hand side of the equation is connected with the average value of the function $\hat{\boldsymbol{\phi}}_1$ in the heterogeneous medium and as such is not dependent on the assumed configuration A of inclusions distribution.

For a small volume fraction v_i the influence of the other inclusions on the selected one may be neglected. Then the conditional ensemble average $\{\cdot\}^*$ appearing in equations (43) and (46) is understood as the ensemble average over all allowable set of the test inclusion shapes and orientations. In order to calculate $\hat{\boldsymbol{\phi}}_1$ the third term on the left-hand side of equation (46) was thus abandoned and its solution assumed to be

$$\hat{\boldsymbol{\phi}}_1 = \mathbb{C}\cdot\xi. \quad (47)$$

After introducing this form of function $\hat{\boldsymbol{\phi}}_1$ into equation (46) a linear equation for constant $\mathbb{C}$ has been obtained which, when used together with (47) in definition of $\boldsymbol{\lambda}_{ef}$, leads to the following expression for the effective thermal conductivity of the composite

$$\boldsymbol{\lambda}_{ef}/\lambda_m = \mathbb{1} + \sigma_i' v_i\{(\mathbb{1}+\sigma_i'\mathbb{P})^{-1}\}^* \cdot [\mathbb{1} - \sigma_i' v_i\{\mathbb{P}\cdot(\mathbb{1}+\sigma_i'\mathbb{P})^{-1}\}^*]^{-1} \quad (48)$$

where $\sigma_i' = (\lambda_i - \lambda_m)/\lambda_m$ and $\mathbb{P}$ is a tensor related to the shape of the test inclusion

$$\mathbb{P} = \lambda_m \int_A \nabla_1 G_x\,\mathbf{n}\,\mathrm{d}A. \quad (49)$$

This tensor has some characteristic features, for example its trace is equal to unity, and has been more deeply discussed elsewhere (e.g. in ref. [19]).

The effective thermal conductivity λ_{ef} of two-component heterogeneous medium with an ellipsoidal shape of inclusions (ellipsoids of revolution) and their random orientation in space is shown in Fig. 3. It has been presented in relation to the effective thermal conductivity of spherical inclusions ($\mathbb{P} = (1/3)\mathbb{1}$) in order to clearly illustrate the influence of inclusion shape on this property of the composite. It can be observed that for this type of the statistically isotropic heterogeneous material and $\lambda_i/\lambda_m > 1$ the effective thermal conductivity may be substantially greater (several times) then the effective thermal conductivity of a composite with spherical shape of inclusions. For $\lambda_i/\lambda_m < 1$, however, a certain asymmetry in λ_{ef} behaviour comes out. For inclusion in the form of the flattened ellipsoids the effective thermal conductivity strongly decreases when the ratio a/b of the characteristic ellipsoid dimensions diminishes, however, for the elongated shape of the inclusion (assuming the form of fibres) and $\lambda_i/\lambda_m < 1$ the value of λ_{ef} is practically indistinguishable from the appropriate value for a composite with spherical inclusions.

The additional comparison between the effective thermal conductivity values obtained from the derived formula (48) and the experimental ones, given in ref. [25] by Yamada and Otta, is shown in Fig. 4. The composite had been made by randomly spreading lead inclusions of parallepipedal shape in the epoxy–resin matrix. Noting the simplifying assumptions that had been made the consistence of the results seems satisfactory.

CONCLUSIONS

A theory of macroscopic, effective description of heat conduction in heterogeneous materials was derived with the help of the ensemble averaging technique and Green's function method. The main results are contained in equations (22) and (23) of the derivation where the relations between the averaged heat flux or energy density and the ensemble averaged temperature are given. These equations are the solutions of the closure problem for the effective heat conduction in non-homogeneous media and allow one to study many problems, both steady and transient.

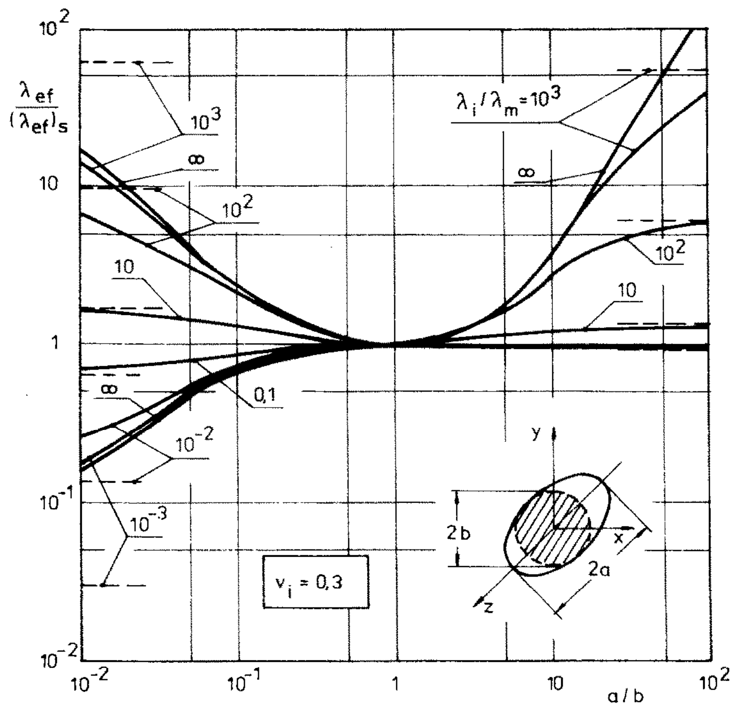

FIG. 3. The effective thermal conductivity λ_{ef} for the statistically isotropic medium with ellipsoidal inclusions spread randomly in a matrix vs ratio of the ellipsoid dimensions. $(\lambda_{ef})_s$ is the effective thermal conductivity of the composite with spherical inclusions.

Far from the boundaries of the medium these equations may be treated as the constitutive relations and the material functions appearing in them as the effective properties of the non-homogeneous material. All of these effective properties were defined with use of the so-called microstructure functions which depend solely on the thermal properties of constituents and their distribution in the material. The discussed constitutive equations are of the nonlocal type and indicate that the effective reaction of a heterogeneous medium varies according to the change in the ratio of a characteristic microdimension l of its structure to a macroscopic dimension L related to variation of the ensemble averaged temperature field in the medium. In the limit of $l/L \to 0$ the heterogeneous medium behaves, in the macroscopic sense, as a quasi-homogeneous one and its effective properties become independent of the microdimension l. This is consistent with general observations of Kunin [23]. Also in this special case the microgeometry of the heterogeneous medium seems to have the greatest influence on its effective thermal conductivity. On the basis of the presented theory many other interesting problems associated with the heat conduction process in heterogeneous media can be studied in depth. Some examples were given in the work.

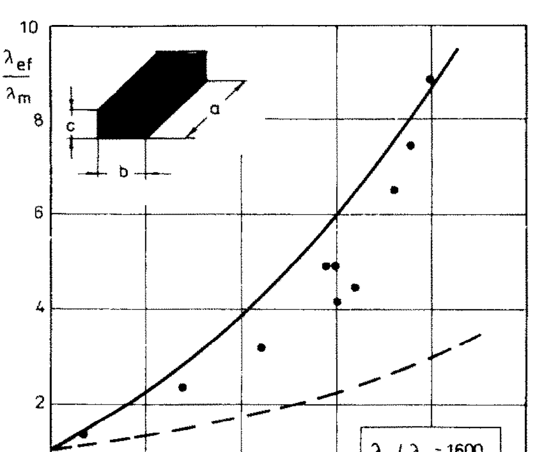

FIG. 4. The effective thermal conductivity of a composite with randomly oriented parallepipedal inclusions made of lead (dimensions $a = 2.55$ mm, $b = 0.8$ mm, $c = 0.45$ mm) distributed in an epoxy matrix vs volume fraction of the inclusions (··· experimental data from Yamada and Otta [25], ——— results obtained from formula (49) [19], --- results obtained from Maxwell formula [22] for the spherical inclusions).

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DESCRIPTION EFFECTIVE, MACROSCOPIQUE DE LA CONDUCTION THERMIQUE DANS LES MATERIAUX HETEROGENES

Résumé—On étude le compartement macroscopique les milleux heterogenes à l'aide de la methode homogenise statistique par l'ensemble des confidurations de la structure. Nonlocales constitutives relations pour la conduction thermique sont déduire. Ces relations se joindent le flux moyenne de chaleur et densité de l'énergie moyenne avec le température moyenne d'un millieu. Toutes les propriétes effective dans les relations sont défine au moyen de noveux proposé, si nommé, microstructures fonctions. Le compartement limite des millieux heterogene pour lentement variables champs thermique sont envisagé. On présente possible application de la théorie in deux exemples ce qui les résultats analytique sont simplement obtenue.

EFFEKTIVEN MAKROSKOPISCHEN BESCHREIBUNG DER WÄRMELEITUNG IN HETEROGENEN MEDIEN

Zusammenfassung—Makroskopischen Verhaltens des inhomogenes Mediums wird untersucht mit Hilfe der statistischen Vermittlerung über das Ensemble der Konfigurationen der inneren Strukturen des Körpers. Ermittelt werden die nonlokalen konstitutive Gleichungen für die Wärmeleitung in diesem Medium. Die Gleichungen verbinden die mittlere Wärme Flux und die mittlere innere Energie mit die mittlere Temperatur des Mediums. Alle effektiven Eigenschaften das sind in diesen Gleichungen betrachtet, werden mit Hilfe die neue geführtent, so gennant Mikrostrukture Funktionen definiziert. Die asymptotische Verhaltens des heterogenen Körpers für langsam wechselnd mittleren Temperatur Felder wird untersucht. Möglische Anwendung für die Theorie wird angegaben mit zwei Bei spiele die sich einfach analytisch lösen lassen.

ЭФФЕКТИВНОЕ МАКРОСКОПИЧЕСКОЕ ОПИСАНИЕ ТЕПЛОПРОВОДНОСТИ В ГЕТЕРОГЕННЫХ МАТЕРИАЛАХ

Аннотация—С использованием метода усреднения по ансамблю исследуются макроскопические тепловые характеристики гетерогенных материалов. Получены соотношения для теплопроводности, связывающие усредненный тепловой поток и плотность энергии с усредненной температурой среды. Все входящие в соотношения эффективные характеристики определяются с помощью новых так называемых микроструктурных функций. Исследуются асимптотические свойства гетерогенных сред при медленно изменяющихся полях средних температур. Приводятся два примера применения теории, когда можно легко получить аналитические результаты.